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CORRECTED BOUNDARY-INTEGRAL EQUATIONS IN PLANAR THERMOELASTOPLASTICITY

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Abstract—There are some errors in the direct boundary-integral equation formulation in terms of displacements or velocities, of problems in planar elastoplasticity and thermoelastoplasticity that have been recently reported in the literature. In particular, lack of proper care in reducing the correctly formulated three dimensional problem to the case of plane strain has resulted in incorrect expressions for certain kernels that appear in the integral equations. A correct direct boundary-integral equation formulation for the plane strain problem in thermoelastoplasticity is presented in this paper.

Several researchers [1–6] have used the boundary-integral equation method to solve problems in solid mechanics including elastic, plastic and, in some cases thermal strains. In these problems it is assumed that the total strain rate $\dot{\epsilon}_{ij}$ can be decomposed according to the equation

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^{e}_{ij} + \dot{\epsilon}^{p}_{ij} + \dot{\epsilon}^{T}_{ij} \tag{1}$$

where $\dot{\epsilon}^{e}_{ij}$ is related to the stress rate $\dot{\sigma}_{ij}$ by Hooke's law, $\dot{\epsilon}^{p}_{ij}$ is the plastic strain rate and $\dot{\epsilon}^{T}_{ij} = \alpha\dot{T}\delta_{ij}$ is the thermal strain rate. Here α is the coefficient of linear thermal expansion, T is the temperature, δ_{ij} the Kronicker delta and, for a typical variable, $\dot{q} = (\partial q/\partial t) = (\mathrm{d}q/\mathrm{d}t)$ where t is time. The total strain rate is related to the displacement rate $\dot{u}_i$ by the kinematic relation

$$\dot{\epsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2. \tag{2}$$

The plastic strain is usually assumed deviatoric so that

$$\dot{\epsilon}^{p}_{kk} = \dot{\epsilon}^{p}_{11} + \dot{\epsilon}^{p}_{22} + \dot{\epsilon}^{p}_{33} = 0. \tag{3}$$

Combining (1), (2) and (3) with Hooke's law and the equations of equilibrium, we can write Navier's equations for the three dimensional problem as

$$\dot{u}_{i,jj} + \frac{1}{1-2\nu}\dot{u}_{k,ki} = -\frac{\dot{F}_i}{G} + 2\dot{\epsilon}^{p}_{ij,j} + \frac{2(1+\nu)}{1-2\nu}\alpha\dot{T}_{,i}. \tag{4}$$

where F_i is the prescribed body force per unit volume, ν is Poisson's ratio and G the shear modulus. The boundary conditions usually comprise prescribed displacements on some parts of the surface and prescribed tractions on the rest although mixed-mixed problems can also be considered.

The boundary-integral formulation for the three dimensional problem gives a solution of (4) in the form [1, 2]

$$\dot{u}_i(p) = \int_S [(U_{ij}(p,Q)\dot{\tau}_j(Q) - T_{ij}(p,Q)\dot{u}_j(Q)]\,\mathrm{d}S_Q + \int_V U_{ij}(p,q)\dot{F}_j(q)\,\mathrm{d}V_q + \int_V \Sigma_{jki}(p,q)[\dot{\epsilon}^{p}_{jk}(q) + \delta_{jk}\alpha\dot{T}(q)]\,\mathrm{d}V_q \tag{5}$$

where τ_j is the traction vector, V is the volume of the body, S its surface, and P and Q are surface points and p and q are interior points respectively. The kernels U_{ij}, T_{ij} and Σ_{jki} are obtained from the solution to Kelvin's problem of a point load in an infinite elastic body and are given in Refs. [1, 2] and other papers, Please note that eqn (5) and the definition of Σ_{jkl} in this paper conform to Mendelson [2].

The stress rates are obtained by differentiating (5) at a load point and using

$$\dot{\sigma}_{ij} = G(\dot{u}_{i,j} + \dot{u}_{j,i}) + \frac{2G\nu}{1-2\nu}\dot{u}_{k,k}\delta_{ij} - 2\left(G\dot{\epsilon}^p_{ij} + G\left(\frac{1+\nu}{1-2\nu}\right)\alpha\dot{T}\delta_{ij}\right). \tag{6}$$

The result is

$$\begin{aligned}\dot{\sigma}_{ij}(p) = &\int_S [V_{ijk}(p,Q)\,\dot{\tau}_k(Q) - T_{ijk}(p,Q)\dot{u}_k(Q)]\,\mathrm{d}S_Q \\ &+ \int_V V_{ijk}(p,q)\dot{F}_k(q)\,\mathrm{d}V_q - 2G\dot{\epsilon}^p_{ij}(p) - 3K\alpha\dot{T}(p)\delta_{ij} \\ &+ \int_V \Sigma_{ijkl}(p,q)[\dot{\epsilon}^p_{kl}(q) + \delta_{kl}\alpha\dot{T}(q)]\,\mathrm{d}V_q\end{aligned} \tag{7}$$

where K is the bulk modulus

$$V_{ijk} = -\Sigma_{ijk} \tag{8}$$

$$\begin{aligned}\Sigma_{ijkl} = &\frac{G}{4\pi(1-\nu)s^3}[3(1-2\nu)(\delta_{ij}s_{,k}s_{,l} + \delta_{kl}s_{,i}s_{,j}) \\ &+ 3\nu(\delta_{il}s_{,j}s_{,k} + \delta_{jk}s_{,l}s_{,i} + \delta_{ik}s_{,l}s_{,j} + \delta_{jl}s_{,i}s_{,k}) - 15s_{,i}s_{,j}s_{,k}s_{,l} \\ &+ (1-2\nu)(\delta_{ik}\delta_{lj} + \delta_{jk}\delta_{li}) - (1-4\nu)\delta_{ij}\delta_{kl}]\end{aligned} \tag{9}$$

$$T_{ijk} = \Sigma_{ijkl}n_l \tag{10}$$

s is the distance between two points p and q, $s_{,i} = (\partial s/\partial x_i|_q)$, the differential at a field point and n_i is the outward unit normal to the surface S. In all the above formulae, the range of subscripts, of course, is 1, 2, 3.

What we have presented so far here is well known. The difficulty arises, however, when we want to determine analogous formulae for plane strain ($\epsilon_{zz} = 0$). It is here that there are errors in the literature [2–4] and it is the purpose of this paper to point these out and correct them.

The case of plane strain ($\epsilon_{zz} = 0$)

Navier's equations for the displacement rates have the same form as (4) with $i, j, k = 1, 2$. The boundary-integral formula analogous to (5), however, now becomes

$$\begin{aligned}\dot{u}_i(p) = &\int_C [U_{ij}(p,Q)\dot{\tau}_j(Q) - T_{ij}(p,Q)\dot{u}_j(Q)]\,\mathrm{d}C_Q + \int_A U_{ij}(p,q)\dot{F}_j(q)\,\mathrm{d}A_q \\ &+ \int_A [\hat{\Sigma}_{jki}(p,q)\dot{\epsilon}^p_{jk}(q) + \hat{\hat{\Sigma}}_{jki}(p,q)\delta_{jk}\alpha\dot{T}(q)]\,\mathrm{d}A_q \qquad (i,j,k = 1,2)\end{aligned} \tag{11}$$

where A is the area of cross section of the body, C the boundary of A, and the kernels are given by

$$U_{ij} = -\frac{1}{8\pi(1-\nu)G}[(3-4\nu)\ln s\,\delta_{ij} - s_{,i}s_{,j}] \tag{12}$$

$$T_{ij} = -\frac{1}{4\pi(1-\nu)s}\left[[(1-2\nu)\delta_{ij} + 2s_{,i}s_{,j}]\frac{\partial s}{\partial n} + (1-2\nu)(s_{,j}n_i - s_{,i}n_j)\right] \tag{13}$$

$$\hat{\Sigma}_{jki} = -\frac{1}{4\pi(1-\nu)s}[(1-2\nu)(\delta_{ij}s_{,k} + \delta_{ki}s_{,j}) - \delta_{jk}s_{,i} + 2s_{,i}s_{,j}s_{,k}] \tag{14}$$

$$\hat{\hat{\Sigma}}_{jki} = -\frac{1}{4\pi(1-\nu)s}[(1-2\nu)(\delta_{ij}s_{,k} + \delta_{ki}s_{,j}) - (1-3\nu)\delta_{jk}s_{,i} + 2s_{,i}s_{,j}s_{,k}]. \tag{15}$$

Also, let us note that the kernel used in [2] from Kelvin's solution is

$$\Sigma_{jki} = -\frac{1}{4\pi(1-\nu)s}[(1-2\nu)(\delta_{ij}s_{,k} + \delta_{ki}s_{,j} - \delta_{jk}s_{,i}) + 2s_{,i}s_{,j}s_{,k}]. \tag{16}$$

This can be proved in two ways.

Proof one. Proceeding as in Mendelson[2] we can show that at an interior point

$$\dot{u}_i = \int_C (U_{ij}\dot{\tau}_j - T_{ij}\dot{u}_j)\,\mathrm{d}C + \int_A U_{ij}\dot{F}_j\,\mathrm{d}A$$
$$+ \int_A \left[2GU_{ij,k}\dot{\epsilon}^p_{jk} + \frac{2G(1+\nu)}{(1-2\nu)}U_{ij,j}\alpha\dot{T}\right]\mathrm{d}A. \tag{17}$$

Using the symmetry of plastic strain

$$2GU_{ij,k}\dot{\epsilon}^p_{jk} = 2G\left[\frac{1}{2}(U_{ij,k} + U_{ik,j})\right]\dot{\epsilon}^p_{jk}$$

and since δ_{jk} is symmetric and $\delta_{jk}\delta_{jk} = 2$

$$2G\frac{(1+\nu)}{(1-2\nu)}U_{ij,j} = 2G\left[\frac{1}{2}(U_{ij,k} + U_{ik,j}) + \frac{3\nu}{2(1-2\nu)}U_{il,l}\delta_{jk}\right]\delta_{jk}.$$

Recalling from Kelvin's elastic solution the equation

$$\Sigma_{jki} = 2G\left[\frac{1}{2}(U_{ij,k} + U_{ik,j}) + \left(\frac{\nu}{1-2\nu}\right)U_{il,l}\delta_{jk}\right]$$

and defining the coefficients of $\dot{\epsilon}^p_{jk}$ and δ_{jk} in the above equations as $\hat{\Sigma}_{jki}$ and $\hat{\hat{\Sigma}}_{jki}$ respectively, we obtain

$$\hat{\Sigma}_{jki} = \Sigma_{jki} - \frac{2G\nu}{(1-2\nu)}U_{il,l}\delta_{jk}$$
$$\hat{\hat{\Sigma}}_{jki} = \Sigma_{jki} + \frac{G\nu}{(1-2\nu)}U_{il,l}\delta_{jk}. \tag{18}$$

Some further algebraic manipulation gives (14) and (15).

Note that the contribution of $\Sigma_{jki} - \hat{\Sigma}_{jki}$ in (11) does not vanish, since, for plane strain, $\dot{\epsilon}^p_{jk}\delta_{jk} = \dot{\epsilon}^p_{11} + \dot{\epsilon}^p_{22} \neq 0$.

Proof two. This follows the method outlined by Swedlow *et al.*[1] and by Mendelson in the Appendix of [2].

Swedlow *et al.*[1] note correctly that for the *three-dimensional problem*

$$\int_{V-V_\epsilon} \sigma^*_{ij}\dot{\epsilon}^e_{ij}\,\mathrm{d}V = \int_{V-V_\epsilon} \epsilon^*_{ij}\dot{\sigma}_{ij}\,\mathrm{d}V \qquad (i,j, = 1,2,3) \tag{20}$$

where the starred fields refer to Kelvin's elastic solution and V_ϵ is a small sphere surrounding the load point p. For the plane strain problem, however, even though (6) is true with $i, j = 1, 2$; (20) is no longer true. This is because, using (1), (2) and (6) we can immediately prove that

$$\dot{\sigma}_{ij} = 2G\dot{\epsilon}^e_{ij} + \frac{2G\nu}{1-2\nu}\dot{\epsilon}^e_{kk}\sigma_{i,j} + \left(\frac{2G\nu}{1-2\nu}\right)(\dot{\epsilon}^p_{kk} - \alpha\dot{T})\delta_{ij} \qquad (i,j,k = 1,2) \tag{21}$$

and of course the starred field, which is the solution of an elastic problem, satisfies a similar equation with the last term in the right of (21) set to zero. Thus, (20) assumes the new form

$$\int_{A-A_\epsilon} \sigma^*_{ij}\dot{\epsilon}^e_{ij}\,\mathrm{d}A = \int_{A-A_\epsilon} \epsilon^*_{ij}\dot{\sigma}_{ij}\,\mathrm{d}A - \left(\frac{2G\nu}{1-2\nu}\right)\int_{A-A_\epsilon} (\dot{\epsilon}^p_{kk} - \alpha\dot{T})\epsilon^*_{ll}\,\mathrm{d}A \qquad (i,j,k,l=1,2). \tag{22}$$

The last term in (22) can be written as

$$-\left(\frac{2G\nu}{1-2\nu}\right)e_k\int_{A-A_\epsilon} U_{kl,l}\dot{\epsilon}^p_{ij}\delta_{ij}\,\mathrm{d}A + \left(\frac{G\nu}{1-2\nu}\right)e_k\int_{A-A_\epsilon} U_{kl,l}\alpha\dot{T}\delta_{ij}\delta_{ij}$$

where e_k represents a traid of orthogonal unit vectors at the load point in the Kelvin problem along which directions loads are applied.

When these terms are included in Mendelson's proof in [2], we obtain (11) with the appropriate kernels given by (18) and (19).

The stress rates are obtained by direct differentiation of (11) resulting in

$$\begin{aligned}\dot{\sigma}_{ij}(p) = &\int_C [V_{ijk}(p,Q)\dot{\tau}_k(Q) - T_{ijk}(p,Q)\dot{u}_k(Q)]\,\mathrm{d}C_Q \\ &+ \int_A V_{ijk}(p,q)\dot{F}_k(q)\,\mathrm{d}A_q - 2G\dot{\epsilon}^p_{ij}(p) - 3K\alpha\dot{T}(p)\delta_{ij} \\ &+ \int_A \hat{\Sigma}_{ijkl}(p,q)\dot{\epsilon}^p_{kl}(q)\,\mathrm{d}A_q + \int_A \hat{\hat{\Sigma}}_{ijkl}(p,q)\delta_{kl}\alpha\dot{T}(q)\,\mathrm{d}A_q \qquad (i,j,k,l=1,2)\end{aligned} \tag{23}$$

where

$$V_{ijk} = -\Sigma_{ijk} \tag{24}$$

$$T_{ijk} = \Sigma_{ijkl}n_l \tag{25}$$

$$\hat{\Sigma}_{ijkl} = \Sigma_{ijkl} + \frac{G}{2\pi(1-\nu)s^2}\{4\nu s_{,i}s_{,j}\delta_{kl} - 2\nu\delta_{ij}\delta_{kl}\} \tag{26}$$

$$\hat{\hat{\Sigma}}_{ijkl} = \Sigma_{ijkl} + \frac{G}{2\pi(1-\nu)s^2}\{-2\nu s_{,i}s_{,j}\delta_{kl} + \nu\delta_{ij}\delta_{kl}\} \tag{27}$$

$$\begin{aligned}\Sigma_{ijkl} = &\frac{G}{2\pi(1-\nu)s^2}[2(1-2\nu)(\delta_{ij}s_{,k}s_{,l} + \delta_{kl}s_{,i}s_{,j}) \\ &+ 2\nu(\delta_{li}s_{,j}s_{,k} + \delta_{jk}s_{,l}s_{,i} + \delta_{ik}s_{,l}s_{,j} + \delta_{jl}s_{,i}s_{,k}) - 8s_{,i}s_{,j}s_{,k}s_{,l} \\ &+ (1-2\nu)(\delta_{ik}\delta_{lj} + \delta_{jk}\delta_{li}) - (1-4\nu)\delta_{ij}\delta_{kl}].\end{aligned} \tag{28}$$

The case of plane stress ($\sigma_{zz} = 0$)

The case of plane stress is included here for completeness. Navier's equation for displacement rate for plane stress has the form [2]

$$\dot{u}_{i,jj} + \left(\frac{1+\nu}{1-\nu}\right)\dot{u}_{k,ki} = -\frac{\dot{F}_i}{G} + 2\dot{\epsilon}^p_{ij,j} + \frac{2\nu}{1-\nu}\dot{\epsilon}^p_{kk,i} + \frac{2(1+\nu)}{(1-\nu)}\alpha\dot{T}_{,i} \qquad (i,j,k=1,2). \tag{29}$$

The solution for the displacement rate has the same form as (5) (with the range of subscripts 1, 2) and the kernels U_{ij}, T_{ij} and Σ_{jki} are given by (12), (13) and (16) respectively with ν replaced by $\bar{\nu} = \nu/(1+\nu)$.

The stress rate equation assumes the form

$$\begin{aligned}\dot{\sigma}_{ij}(p) = &\int_C [V_{ijk}(p,Q)\dot{\tau}_k(Q) - T_{ijk}(p,Q)\dot{u}_k(Q)]\,\mathrm{d}C_Q + \int_A V_{ijk}(p,q)\dot{F}_k(q)\,\mathrm{d}A_q \\ &+ \int_A \Sigma_{ijkl}(p,q)[\dot{\epsilon}^p_{kl}(q) + \delta_{kl}\alpha\dot{T}(q)]\,\mathrm{d}A_q \\ &- 2G\dot{\epsilon}^p_{ij}(p) - \left(\frac{2G\bar{\nu}}{1-2\bar{\nu}}\right)\dot{\epsilon}^p_{ll}(p)\delta_{ij} - \left(\frac{2G}{1-2\bar{\nu}}\right)\alpha\dot{T}(p)\delta_{ij} \qquad (i,j,k,l=1,2)\end{aligned} \tag{30}$$

where the kernel Σ_{ijkl} has the same form as (28) with ν replaced by $\bar{\nu}$ and, as before,

$$V_{ijk} = -\Sigma_{ijk} \tag{31}$$

$$T_{ijk} = \Sigma_{ijkl} n_l. \tag{32}$$

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